A Local Realistic Reconciliation of the Einstein-Podolsky-Rosen Paradox.

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The exact violation of Bell's Inequalities is obtained with a local realistic model for spin. The model treats one particle that comprises a quantum ensemble and simulates the EPR data one coincidence at a time as a product state. When isolated, spin is assumed to have two dimensional structure formed from two orthogonal axes of spin quantization. These two components have hidden states which account for the quantum correlation between EPR pairs without violation of local causality. The model futher predicts the filter angles that maximize the spin correlation and give new insignt into Bell's Inequalities and the Heisenberg Uncertainty Principle.

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INTRODUCTION

Although quantum mechanics is successful in describing almost all observed microscopic events, it has wide and varied interpretations [1]. Of these, the most commonly accepted today is the Copenhagen, see *e.g.* [2]. This and most of the others, [3], [4], [5], to mention only a few, assume quantum is a complete theory with no deeper level. Only Bohmian mechanics, [6] and the statistical ensemble interpretation, [7], [8] consider quantum mechanics to be incomplete. Part of the motivation for a deeper theory is that quantum mechanics fails to describe individual events (for example it cannot predict how a particle with spin randomly ejected from a source will be deflected in a Stern-Gerlach filter). Although the completeness of quantum theory has been questioned since its inception, it was in 1935 that EPR [9] succinctly made their case: position and momentum are simultaneous elements of physical reality but quantum theory cannot describe them both simultaneously, as a physical theory should. The same conclusion can be drawn for any two non-commuting physical operators, like two components of the Pauli spin operator for a spin $\frac{1}{2}$. Using spin rather than position and momentum simplifies not only the theoretical treatment but also the experiments [6].

The statistical ensemble interpretation considers a quantum state to be an ensemble average over many particles that carry additional variables that lie beyond our ability to measure. These Local Hidden Variables (LHV) are expected to describe the individual particles as real (ontological) entities that evolve deterministically. They collectively make up the quantum state. Bell's theorem [10], [11] is usually interpreted as ruling out any such variables. Bell, however, had a different view of his theorem and argued that the violation of his inequalities demonstrated a conflict between the predictions of quantum mechanics and the concept of local causality from Special Relativity, see *e.g.* [12].

The approach taken here is quite different. Rather than interpreting the violation of Bell's Inequalities as being a result of the breakdown of local causality, it is found that the act of measurement is the culprit. Quantum mechanics is a theory of measurement and the act of observation destroys polarization that exists in the absence of a measuring probe. Bell considered that elements of physical reality could exist in the absence of observation and introduced the concept of beables [13] to contrast observables. In this paper it is found that a spin $\frac{1}{2}$ admits such beables which, upon measurement, become the usual point particle spin with two states as observed in a Stern-Gerlach experiment. However in the absence of measurement, a spin is assumed to have structure formed from two orthogonal axes of quantization [14]. The extra polarization carried by a spin is shown here to account for the violation of Bell's Inequalities rather than the breakdown of local causality.

If, however, Special Relativity is accepted as correct, which is the view here, and additionally that quantum mechanics is a complete physical theory, which is not the view here, then Nature must not only be statistical but also non-local. Although widely accepted, nonetheless, the former goes against our basic ontological view of Nature while the latter defies a logical explanation. Motivated by this, many attempts have been made to show that Bell's inequalities are mathematically at fault, see *e.g.* [15], or suggest loopholes in the experiments, [16], but these attempts

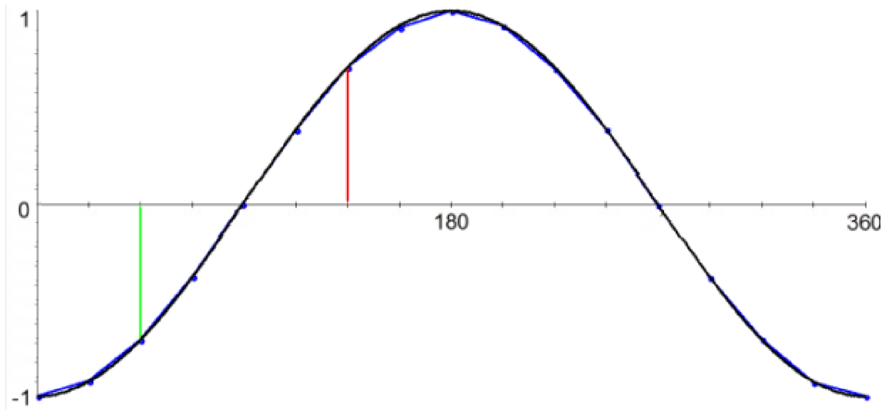

FIG. 1: A local realistic simulation of the EPR data calculated at every 22.5 degrees and the points connected. The smooth curve is $-\cos\theta_{ab}$ plotted as a function of θ_{ab}.

frequently meet with resistance, see e.g. [17], [18]. Others seek to find LHV models that indeed account for the quantum correlation that leads to the violation of Bell's theorem, see *e.g.* [19], [20], [21]. This has also met with resistance, [22]. Although the above is by no means a complete survey of the attempts to resolve the EPR paradox, it underscores the fact that a purely mathematical approach is unlikely to provide acceptable proof that quantum mechanics is incomplete.

An alternate approach is to use computer simulation to test LHV models on an event-by-event basis [23]. That is simulations of EPR experiments can be performed by computing the individual coincidences one by one [24]. Since each coincidence depends upon its unique set of LHV, computational spanning these gives an independent test of a model. This is the approach used here. A Java program [25] simulates [23] the EPR data [26], [27] by taking products of EPR pairs [28] with no connectivity between them. This program has been adapted to the spin model presented here and is available at the web site [24]. The statistical data so generated exactly fits the quantum result for the correlation of $E_{\text{quantum}}(a,b) = -\cos\theta_{ab} = -\mathbf{a}\cdot\mathbf{b}$ where θ_{ab} is the angle between the two filter settings **a** and **b**, see Figure 1.

In this work then, an EPR pair is illustrated by two spins $\frac{1}{2}$ originally in a singlet state. Upon separation an EPR pair remains correlated by its common origin while obeying local causality. Local realism is often misconstrued to mean classical. In this work the spin model uses no classical concepts. Realism means that a particle can occupy only pure dispersion free quantum states at any instant. Locality means Einstein, or E-locality: when a pair of particles is beyond the range of any forces between them, their state factors into a product, and any interactions between particles propagate at or less than the speed of light.

In the model presented here, one of the two axes of spin quantization is directed along the z axis and the other along the x axis defined in a body-fixed coordinate frame of the spin, (x, y, z). In terms of Pauli spin operators, the components are taken as σ_z and σ_x. It is assumed that each axis carries a magnetic moment of equal magnitude μ being the usual literature values for different particles. One axis of quantization carries polarization which, due to the Heisenberg Uncertainty Principle, cannot be measured simultaneously with the other. If one axis is measured, say with a Stern-Gerlach filter, then the states are those usually observed and represented on the unit Bloch sphere. In this case the second axis displays no detectable polarization and its states are fully randomized. Whereas the measured axis is characterized by the usual eigenstates of ±1, the unmeasured axis is represented by quantum coherence terms.

The correlation between EPR pairs is found from quantum theory by assuming the persistence of entanglement [29] as each pair separates. In contrast here, until they encounter a filter, the two EPR particles are considered to be in free flight and are correlated by conservation of linear and angular momentum established by their common origin. One particle approaches a filter at Alice's location oriented in the direction **a** and the other encounters Bob's filter oriented at angle **b**. In the model the states of an individual spin are found of the form $\left|\pm\sqrt{2},\lambda\right\rangle$ where λ denotes the LHV. The simulation starts with these states by calculating one EPR pair at a time. Then by averaging over λ the quantum result is obtained, albeit with some caveats.

An immediate consequence of this spin model is that only half the correlation can be measured in one experiment. This, in turn, means that the model does not violate the CHSH form of Bell's Inequalities, [30]. This is a significant difference from the usual interpretation of the EPR coincidence data. Two simulation are necessary to obtain the full quantum correlation,

$$E_{sim}\left(a,b\right)=-\overset{z-axis}{\frac{1}{2}\cos\theta_{ab}}-\overset{x-axis}{\frac{1}{2}\cos\theta_{ab}}= \tag{1}$$

$$\equiv E_{sim}^{z}\left(a,b\right)+E_{sim}^{x}\left(a,b\right)=-\cos\theta_{ab} \tag{2}$$

This is plotted in Figure 1 along with the simulated data that ran over ten million EPR pairs.

The CHSH form of Bell's Inequalities [30] is given by

$$S=|E(a,b)+E(a,c)+E(d,b)-E(d,c)|\leq 2 \tag{3}$$

The computer simulated data for the four filter angles θ_{ab} (see Figure (5)) that are known to maximize the EPR correlation cannot violate the CHSH equation since only the correlation from one axis, say z, can be measured in one experiment,

$$S^{z}=|E_{sim}^{z}(45^{o})+E_{sim}^{z}(45^{o})+E_{sim}^{z}(45^{o})-E_{sim}^{z}(135^{o})|=\sqrt{2}\leq 2 \tag{4}$$

An identical expression is simulated for the x axis. Only the sum of the two independent simulations gives the full quantum value of $2\sqrt{2}$. Recall the CHSH equation is derived by partitioning hidden variable space into two regions [30] but, as a classical equation, does not take into account Heisenberg Uncertainty. Hence even though the two simulations agree with the predictions of quantum mechanics, $2\sqrt{2}$, from one experiment the CHSH equation is satisfied, $\sqrt{2}<2$. In contrast the spin model does violate Bell's original form [10] in one simulation, $\sqrt{2}\leq 1$, with the conclusion that a product state indeed can violate Bell's Inequalities. This is discussed further in this paper.

The question immediately arises regarding EPR data [31], [27] which appear to violate the CHSH equation in one experiment. Heisenberg again reconciles this because half the coincidences that are present cannot be measured when two complementary axes exist. The correlation is given by,

$$E_{1\ axis}\left(a,b\right)=\frac{N^{++}+N^{--}-N^{+-}-N^{-+}}{N^{++}+N^{--}+N^{+-}+N^{-+}} \tag{5}$$

in terms of the number of coincidences recorded for the four cases of transmission, + and absorption, -, divided by the total number of coincidences N. The validity of this expression is not questioned, but it cannot be used if spin has two orthogonal axes of quantization. In that case each EPR pair can form coincidences from both the Z and X axes so the total correlation must be

$$E_{2\ \mathrm{axes}}\left(a,b\right)=\frac{\left(N_{Z}^{++}+N_{Z}^{--}-N_{Z}^{+-}-N_{Z}^{-+}\right)+\left(N_{X}^{++}+N_{X}^{--}-N_{X}^{+-}-N_{X}^{-+}\right)}{2\left(N_{Z}^{++}+N_{Z}^{--}+N_{Z}^{+-}+N_{Z}^{-+}\right)} \tag{6}$$

The total number of Z and X coincidences can only be $2N$ for N EPR pairs. Finally since coincidences can be detected from either the Z or the X axes, but not both simultaneously, the expression reduces the correlation measured in one experiment to,

$$E_{2\ \mathrm{axes}}\left(a,b\right)=\frac{N^{++}+N^{--}-N^{+-}-N^{-+}}{2\left(N^{++}+N^{--}+N^{+-}+N^{-+}\right)} \tag{7}$$

This means the coincidences from the undetected axis are counterfactual, [32], [33], and with this modification, the computer simulated correlation per axis is in agreement with the experimental data.

Another way of expressing this is to consider a gedanken experiment where, in spite of non-commutation of the two Pauli spin components, both somehow can be measured simultaneously. In that case a single EPR pair would record a coincidence in the Z channels, and simultaneously another coincidence in the X channels. The total number of coicidences for N EPR pairs is again $2N$.

A second experiment performed with the filter angles chosen 90 degrees from the first will be inconclusive because the two axes are indistinguishable, giving identical results for both.

The Fair Sampling Assumption (FSA) [34], [35] does not fail for this model. The FSA states the number of detected EPR pairs is representative of the total number of pairs emitted. Hence any pairs not detected experimentally are assumed to be a result of instrumental inefficiencies. The treatment here assumes 100% efficiency. As a particle approaches a filter, either the z axis or the x axis aligns with the external probe field. Hence every EPR pair is counted. However as one axis aligns, the other randomizes in the plane perpendicular to the probe axis [36], [37].

For those particles emerging from a filter, the usual spin states are observed in the direction of the probe field, $|\pm1\rangle_Z$ and which can be conveniently represented as antinodal points on the Bloch Sphere. Decoherence is a physical manifestation of Heisenberg Uncertainty in this model.

The point of view here is certainly unconventional, and gives a different interpretation of the EPR data based upon the simulation results of this model. In one simulation it is not possible to obtain more than half the correlation, but nonetheless the results are consistent. After the spin model is introduced, in the following section, its eigenstates are obtained which are valid in free flight. This leads to the introduction of two hidden states that can never be measured: the "beable" states. Measurement means the introduction of a probe so the particle is no longer in free flight. It is emphasized here that measurement can fundamentally change a system and destroy properties. In free flight, however, the model naturally leads to "horizontal" and "vertical" components of angular momentum with equal magnetic moments of magnitude $\sqrt{2}\mu$.

The states for each partner of an EPR pair have the same LHV and opposite angular momentum. Using these states the expectation value for + or - events for a filter set in some orientation is simply the projection of the model's states onto the filtered states. For every set of LHV, a spin has four states available to it. Two are associated with the horizontal axis and two with the vertical axis. All four states cannot be simultaneous eigenvalues: if two are represented as diagonal, and correspond to polarization, then the complementary axis is off-diagonal in the same representation, and are quantum coherences. Upon measurement, the quantum coherence terms are randomized which is responsible for the loss of half the correlation, consistent with the Heisenberg relations.

It is impossible to experimentally confirm two axes exist because this means accepting as real counterfactual coincidences. Consistent with the "H" in LHV, the choice of local realism is subjective. Nonetheless, the model gives new results which the usual description of spin cannot explain. For one, the filter settings that maximize the violation of Bell's Inequalities are correctly predicted. Additionally it is shown that the model can violate Bell's original Inequalities, but due to the complementary nature of the two axes, the CHSH form is not violated. In short, the two dimensional spin model reconciles the EPR paradox but requires fundamental changes from the current statistical, indeterministic and non-local view of the microscopic.

THE SPIN $\frac{1}{2}$ MODEL.

Spin $\frac{1}{2}$ is standardly described by the three components of the Pauli spin vector $(\sigma_X, \sigma_Y, \sigma_Z)$ in the laboratory frame $(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}})$ where measurements are performed. Stern-Gerlach experiments filter a beam of particles carrying a spin $\frac{1}{2}$into two pure states defined by the orientation of the magnetic field in, say, the Z direction. Recall that quantum mechanics cannot predict whether a single particle will deflect up or down. Only after a statistically large number of particles have passed does Malus's law emerge [38]. This work treats those particles that make up the quantum ensemble.

The model describes one isolated spin by the operator set $(\sigma_x, i\sigma_y, \sigma_z)$, *i.e.* two orthogonal components of the Pauli spin operator, σ_x and σ_z, which give an individual spin two dimensional flat structure. Its orientation in space is established by the operator, $\sigma_z\sigma_x = i\sigma_y$. Since this spin has structure, a body fixed coordinate frame, $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, defines its orientation. This is related to the laboratory frame by a rotation through angles θ, ϕ which are LHV.

After passing a Stern-Gerlach filter, the statistical state is described by a density operator for spins of $\frac{1}{2}$ magnitude,

$$\rho = \frac{1}{2}\left(I + \mathbf{P}\cdot\sigma\right) \tag{8}$$

The polarization or Bloch vector, $\mathbf{P}$, is taken to have magnitude $|\mathbf{P}| = 1$, required for pure states. This represents the usual case [39]: spins are polarized along one axis of quantization, defined by the laboratory magnetic field after they have traversed the filter. The value of the polarization vector measures the difference between the number of spins deflected up and the number deflected down.

Before reaching a filter, and the particle is in free flight, it is assumed that the state operator, call it A for Alice, has a similar form to the usual statistical state operator, Eq.(8),

$$s^A\left(\mathbf{n}_{n_z,n_x}\right) = \frac{1}{2}\left(I + \mathbf{n}_{n_z,n_x}\cdot\sigma^A\right) \tag{9}$$

but with the Bloch vector replaced by a unit vector defined in the body frame of a particle given by

$$\mathbf{n}_{n_z,n_x} = \frac{1}{\sqrt{2}}\left(n_z\hat{\mathbf{z}}+n_x\hat{\mathbf{x}}\right) \tag{10}$$

Whereas the polarization vector $\mathbf{P}$ is directed so it aligns with some polarizing field, in the case of one spin, Eq.(9), the local surroundings must be isotropic, *i.e.* the particle is isolated. Therefore the two axes, $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$, are indistinguishable and are randomly oriented relative to the laboratory frame.

The integers $(n_z, n_x) = (\pm 1, \pm 1)$ are LHV in addition to θ, ϕ. They correspond to the quadrants of the body fixed frame in the xz plane. By definition, a single real particle can only have pure states. Hence a two dimensional spin can equally occupy any of its body fixed quadrants, but only one at any instant.

The EPR partner usually called Bob has a similar state operator but with angular momentum which is aligned oppositely,

$$s^B\left(\mathbf{n}_{n_z,n_x}\right) = \frac{1}{2}\left(I - \mathbf{n}_{n_z,n_x}\cdot\sigma^B\right) \tag{11}$$

but otherwise Bob has identical LHV to Alice and the two are treated the same *i.e* EPR pairs have identical values of $\mathbf{n}_{n_z,n_x}$.

Although Eq.(11) has the same structure and properties as a statistical density operator, Eq.(8) it differs in that it represents the unique pure states available to one single spin in the absence of measurement. In this sense it is a pure state operator rather than a density operator, and it is assumed that quantum theory can be applied to isolated spins. The states found are Bell's beables [13].

EIGENSTATES OF THE 2D SPIN

Substitution of the Pauli spin matrices into Eq.(9) gives the body frame z representation of the state operator as

$$s^A\left(\mathbf{n}_{n_z,n_x}\right) = \frac{1}{2}\left(I + \frac{1}{\sqrt{2}}\left(n_z\sigma_z+n_x\sigma_x\right)\right) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} n_z & n_x \\ n_x & -n_z \end{pmatrix}\right] \tag{12}$$

In comparison the statistical operator, Eq.(8), can be written in the laboratory Z representation as,

$$\rho = \frac{1}{2}\left(I + \sigma_Z\right) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \tag{13}$$

The identity term, I, corresponds to a completely random unpolarized state. The second term in Eq.(12) shows that the 2D spin has off-diagonal elements and these are absent in the usual density operator expression, Eq.(13). In Eq.(12), the diagonal elements express the polarization associated with the z axis while the off-diagonal terms are quantum coherences that contain the same information regarding the x axis. Both are responsible for the correlation between EPR pairs. In contrast, Nature needs no representation and the two axes exist simultaneously.

The eigenstates of $(n_z\sigma_z+n_x\sigma_x)$ in the even and odd quadrants are $\left|\pm, \sqrt{2}, \lambda\right\rangle \rightarrow \left|\pm, \sqrt{2}, \theta, \phi, n_z, n_x\right\rangle \rightarrow \left|\pm, \sqrt{2}\right\rangle_{n_z,n_x}$ where to simplify the notation, the θ, ϕ LHV will not be displayed, but understood to be present and generally different for each state,

$$\left|\pm, \sqrt{2}\right\rangle_{+1,+1} = \frac{1}{\sqrt{2\mp\sqrt{2}}}\left(|-,x\rangle \pm |-,z\rangle\right) \;;\; \left|\pm, \sqrt{2}\right\rangle_{-1,-1} = \frac{1}{\sqrt{1\pm\sqrt{2}}}\left(|-,x\rangle \mp |-,z\rangle\right) \tag{14}$$

$$\left|\pm, \sqrt{2}\right\rangle_{+1,-1} = \frac{1}{\sqrt{2\mp\sqrt{2}}}\left(|+,x\rangle \mp |-,z\rangle\right) \;;\; \left|\pm, \sqrt{2}\right\rangle_{-1,+1} = \frac{1}{\sqrt{2\pm\sqrt{2}}}\left(|+,x\rangle \pm |-,z\rangle\right) \tag{15}$$

which are superpositions of the states defined in the body fixed frame rather than the usual laboratory frame,

$$|+,z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\;\; |-,z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; |\pm,x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{16}$$

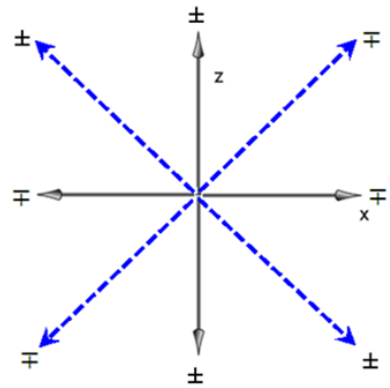

FIG. 2: Spin depicted in an isotropic environment showing the eight possible pure states with two per quadrant. The 2D spin can occupy only one state at any instant. The states bisecting the quadrants are the eigenstates shown in Eqs.(14) and (15).

The eigenvalues of the 2D spin operator are $\pm\sqrt{2}$,

$$(n_z\sigma_z+n_x\sigma_x)\left|\pm,\sqrt{2}\right\rangle_{n_z,n_x} = \pm\sqrt{2}\left|\pm,\sqrt{2}\right\rangle_{n_z,n_x} \tag{17}$$

Figure 2 depicts these hidden states, [14], which bisect each quadrant and, as the sum of two orthogonal unit vectors, have length of $\sqrt{2}$. There are eight possible pure degenerate states for each spin: two pure states per quadrant.

The two states for each quadrant have the symmetry

$$\left|\pm,\sqrt{2}\right\rangle_{+1,+1} = \left|\mp,\sqrt{2}\right\rangle_{-1,-1} \tag{18}$$

$$\left|\pm,\sqrt{2}\right\rangle_{+1,-1} = \left|\mp,\sqrt{2}\right\rangle_{-1,+1} \tag{19}$$

so the results are the same for opposite quadrants, but differ between the odd and even quadrants, thereby giving possible "horizontal" and "vertical" components in the body frame of each spin.

In this LHV theory, superposition occurs only *within* a single spin, and beable states are formed that bisect the two axes of quantization, Figure 2. Superposition cannot be applied between the states in Eqs.(14) and (15) because these are pure states.

When a probe is present, one of the two axes aligns with it, and the other is randomized consistent with the conclusions of Scully, [37], see Figure 3. In other words, half the correlation is destroyed by the act of measurement. In the presence of a vector probe, the states, Eqs.(14) and (15) cannot form, and the usual point particle description of spin is recovered.

This description appeals to a classical view of Nature: that objects are real (pure states); deterministic (based on a spin's LHV its deflection can be predicted before it enters the Stern-Gerlach filter oriented in a known direction); and local (EPR pairs are correlated only by their common origin, see Eqs.(9) and (11)). However there is nothing classical about this model. Other than having different eigenvalues, the degenerate states, $\left|\pm,\sqrt{2}\right\rangle_{n_z,n_x}$ Eqs.(17), are mathematically identical to the usual point particle states $|\pm,z\rangle$: the former can be represented on a Block sphere of radius $\sqrt{2}$ whereas the latter on a Bloch sphere of unit radius. The difference between the two spheres accounts for the extra quantum correlation that causes the violation of Bell's Inequalities.

In Figure 3, the arrows in (b) and (c) depict an individual Bloch vector which collectively align in a magnetic field leading to the observed statistical observable called the magnetic susceptibility. The precession depicted is not classical but rather randomization of the quantum coherences, [37].

Simulating the EPR data suggests that in free flight a spin does not change between its horizontal and vertical axes, see Figure 2 and Eqs.(18) and (19). Otherwise the correlation between Alice and Bob is lost. Using this model, the usual degenerate pair of statistical states in the absence of a field, $|\pm,z\rangle$, are replaced by the purely ontic states, $\left|\pm\sqrt{2},\theta,\phi\right\rangle_{n_z,n_x}$.

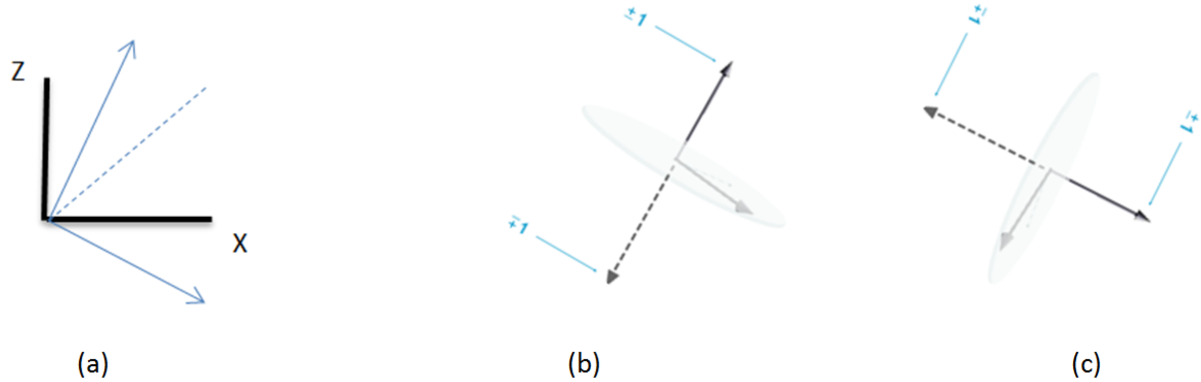

FIG. 3: Depiction of how half the correlation is lost in the presence of a probe: (a) The 2 body-fixed axes of spin quantization, z and x (heavy lines). The two arrows depict two possible orientations of a vector probe field in the laboratory frame, one closer to the z axis and the other closer to the x axis (only one can exist at any instant): (b) The case when the probe is closer to the z axis. Then the 2D spin deterministically evolves until the z axis aligns with the probe while the x component randomize in the XY plane. (c) The same as case (b) but now the probe is closer to the x axis which now aligns in the X direction and the z component randomize in ZY plane.

FILTERING A SPIN

In EPR experiments the particles approach filters which are oriented in direction **a** for Alice's particle and **b** for Bob's. The observables are defined as the pure spin states that result after the filtering.

$$\mathbf{A}_a^{\pm} \equiv \left|\pm, a\right\rangle \left\langle \pm, a\right| \tag{20}$$

These are the usual spin states expressed relative to the laboratory frame by θ, ϕ, which in this case are,

$$\left|+, a\right\rangle = \begin{pmatrix} \cos\left(\theta_a/2\right) e^{-i\phi_a/2} \\ \sin\left(\theta_a/2\right) e^{i\phi_a/2} \end{pmatrix} \; ; \; \left|-, a\right\rangle = \begin{pmatrix} -\sin\left(\theta_a/2\right) e^{-i\varphi_a/2} \\ \cos\left(\theta_a/2\right) e^{i\phi_a/2} \end{pmatrix} \tag{21}$$

The probability for spin-up, transmission and spin-down, absorption are calculated by finding the projection of the pure spin states, $\left|\pm, \sqrt{2}\right\rangle_{n_z, n_x}$, in the direction of the filtered states, Eq.(21). For this the eigenstates in Eqs.(14) and (15) are transformed to the laboratory frame by a rotation of θ, ϕ. Finally the observable, Eq.(20) is expanded using the resolution of the identity,

$$\mathbf{A}_a^{\pm} \equiv \sum_{i,l} \left|i\right\rangle \left\langle i|\pm, a\right\rangle \left\langle \pm, a|j\right\rangle \left\langle j\right| \tag{22}$$

Choosing the states $|i\rangle$ and $|j\rangle$ to be those of the isolated 2D spin, (Eqs.(14) for the even quadrants and (15) for the odd quadrants), four possible states are available to a spin, each depending on its LHV. Whether a particle is transmitted or absorbed is determined by which of its four possible states it occupies,

$$\mathbf{A}_{a,1,1}^{\pm}\left(\theta, \phi\right) = \frac{1}{2}\left(1 \pm a_a^{+-}, \left(\theta, \phi\right)\right) \left|+, \sqrt{2}, \theta, \phi\right\rangle \left\langle +, \sqrt{2}, \theta, \phi\right|_{1,1} \tag{23}$$

$$+ \frac{1}{2}\left(1 \mp a_a^{+-}, \left(\theta, \phi\right)\right) \left|-, \sqrt{2}, \theta, \phi\right\rangle \left\langle -, \sqrt{2}, \theta, \phi\right|_{1,1} \tag{24}$$

$$\pm \frac{1}{2}\left(a_a^{-+}, \left(\theta, \phi\right) + ic_a\left(\phi\right)\right) \left|+, \sqrt{2}, \theta, \phi\right\rangle \left\langle -, \sqrt{2}, \theta, \phi\right|_{1,1} \tag{25}$$

$$\pm \frac{1}{2}\left(a_a^{-+}, \left(\theta, \phi\right) - ic_a\left(\theta\right)\right) \left|-, \sqrt{2}, \theta, \phi\right\rangle_{1,1} \left\langle +, \sqrt{2}, \theta, \phi\right|_{1,1} \tag{26}$$

In order to emphasize that these states for a single spin depend upon its LHV, the angles, θ, ϕ are displayed, but these will be suppressed again from here on.

Equations (23) to (26) show that four states are available to an isolated spin. The diagonal matrix elements give the polarization of the states relative to the filter setting, $\mathbf{a}$,

$$\left|\left\langle \pm, a | +, \sqrt{2}\right\rangle_{1,1}\right|^2 = \frac{1}{2}\left(1 \pm a_a^{+-}\right) \tag{27}$$

$$\left|\left\langle \pm, a | -, \sqrt{2}\right\rangle_{1,1}\right|^2 = \frac{1}{2}\left(1 \mp a_a^{+-}\right) \tag{28}$$

and the off-diagonal quantum coherences are,

$${}_{1,1}\left\langle +, \sqrt{2} | \pm, a\right\rangle \left\langle \pm, a | -, \sqrt{2}\right\rangle_{1,1} = \pm\frac{1}{2}\left(a_a^{-+} + ic_a\right) \tag{29}$$

The coefficients are found by calculating the matrix elements giving,

$$a_a^{\pm\mp} = \frac{1}{\sqrt{2}}\left[(\cos(\theta_a - \theta) \pm \sin(\theta_a - \theta)\cos^2\left(\frac{\phi_a - \phi}{2}\right) + (\cos(\theta_a + \theta) \mp \sin(\theta_a + \theta)\sin\left(\frac{\phi_a - \phi}{2}\right)\right] \tag{30}$$

$$c_a = \sin\theta_a \sin\left(\phi_a - \phi\right) \tag{31}$$

The treatment for the odd quadrants gives identical results upon interchanging $a^{+-} \longleftrightarrow a^{-+}$.

The diagonal coefficients of Eqs.(23) and (24) give the exact chance (no dispersion) a spin has of being filtered or absorbed. The off-diagonal quantum coherences Eqs.(25) and (26) contain the same information about its perpendicular component. These are as real as the diagonal eigenstates but are impossible to detect.

Equations (23) to (26) can be organized into a two by two matrix in the basis states of the spin in the Laboratory frame, $\left|\pm, \sqrt{2}\right\rangle_{n_z, n_z}$ of Eqs.(14) and (15), which for the 1,1 quadrant can be expressed as,

$$\mathbf{A}_{a,+1,+1}^{\pm} = \frac{1}{2}\begin{pmatrix} 1 \pm a_a^{+-} & \pm a_a^{-+} \pm ic_a \\ \pm a_a^{-+} \mp ic_a & 1 \mp a_a^{+-} \end{pmatrix} \tag{32}$$

which is identical to quadrant $-1, -1$. The results for the odd quadrants, $A_{a,\pm1,\mp1}^{\pm}$ are found by interchanging: $a_a^{+-} \leftrightarrow a_a^{-+}$. It is emphasized that no further diagonalization is possible because the pure states cannot superpose.

SIMULATION OF EPR CORRELATIONS

After transforming to the laboratory frame Z, the two states in the first quadrant are

$$\mathbf{A}_{a,1,1}^{\pm} = \frac{1}{2}\left(I \pm \left(a_a^{+-}\sigma_Z + a_a^{-+}\sigma_X + c_a\sigma_Y\right)\right) \tag{33}$$

Since the three Pauli spin operators do not commute they cannot simultaneously be measured. For this reason, the calculation ignores the off-diagonal terms leaving only,

$$\mathbf{P}_{a,1,1}^{\pm} \equiv \frac{1}{2}\begin{pmatrix} (1 \pm a_a^{+-}) & 0 \\ 0 & (1 \mp a_a^{+-}) \end{pmatrix} \tag{34}$$

This describes the probability that a $\pm$ event will occur when Alice's spin is in one of its two states, Eqs.(14) and (15). Bob's spin has a similar form,

$$\mathbf{P}_{b,1,1}^{\mp} \equiv \frac{1}{2}\begin{pmatrix} \left(1 \mp a_b^{+-}\right) & 0 \\ 0 & \left(1 \pm a_b^{+-}\right) \end{pmatrix} \tag{35}$$

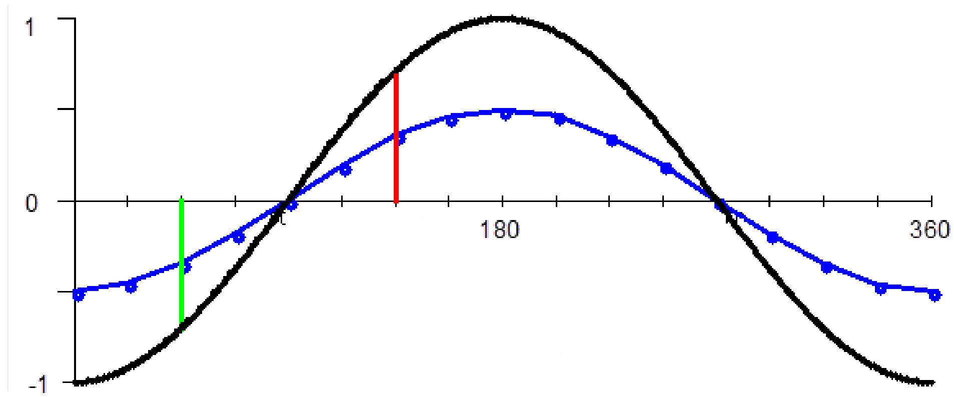

FIG. 4: The result of simulating the EPR correlation for one axis of quantization. The quantum coherences are ignored and the correlation from one axis gives half the correlation predicted from quantum mechanics.

with the same LHV but different filter angle, **b**. In order to satisfy conservation of angular momentum the only combinations that can form EPR pairs must have opposite signs,

$$P_a^{\pm} P_b^{\mp} = \frac{1}{4}\left(1 \pm a_a^{+-}\right)\left(1 \mp a_b^{+-}\right) \tag{36}$$

Again the treatment for the odd quadrants is identical when the coefficients are interchanged, $a_a^{+-} \leftrightarrow a_a^{-+}$.

The simulation is done with a Java computer program written by Chantal Roth [25] which ensures that locality is not violated and calculates each EPR coincidence as a product from Eq.(36). The program was modified to account for Heisenberg by calculating each axis separately and summing the two. The LHV were randomly chosen for each of the ten million simulated coincidences except for the azimuthal angle ϕ which is set to zero (this physically means that the linear momentum is perpendicular to the angular momentum for each spin). The filter angles for Alice and Bob are chosen independently of each other.

Sixteen angles, each differing by 22.5^o were used for the filter settings, θ_{ab}, between Alice and Bob. For each EPR coincidence associated with the Z component, Eqs.(34) and (35), the exact dispersion free probability of a plus or minus outcome is calculated and combined with its partner, Eq.(36), thereby forming a coincidence in one of the four possibilities, (++),(–),(+-),(-+). The numbers of coincidences counted are accumulated separately into bins. Since the off-diagonal elements are ignored, the correlation obtained is from one axis only,

$$E_{1\ axis}\left(a,b\right) = \left(P_a^{+} - P_a^{-}\right)\left(P_b^{+} - P_b^{-}\right) \tag{37}$$

$$= \frac{N^{++} + N^{--} - N^{+-} - N^{-+}}{N^{++} + N^{--} + N^{+-} + N^{-+}} \tag{38}$$

and each coincidence randomly has different LHV. This leads to the ensemble displayed in Figure 4, and corresponds to the correlation from one axis. The simulated smaller curve fits to half the quantum correlation,

$$E_{1\ \mathrm{axis}}\left(a,b\right) = -\frac{1}{2}\cos\theta_{ab} \tag{39}$$

COMPLEMENTARY STATES

Although it is possible to ignore the quantum coherences in the simulation so that the correlation obtained is due to only one axis, experimentally this is not possible. It is, however, possible to theoretically change representations so the quantum coherences are displayed as polarization. A change in representation means doing a different experiment by shifting the external field to point along other directions, in this cases along the Z direction in the Z representation or along the X for the X representation, leading to Eq.(32) represented equally well in either case by,

$$\mathbf{A}^{\pm}_{a,1,1} = \sum_{m,m'=-1}^{+1} |m, Z\rangle_{1,1} A^{Z}_{m,m'} \langle m', Z|_{1,1} = \frac{1}{2}\begin{pmatrix} 1 \pm a_a^{+-} & \pm a_a^{-+} \pm ic_a \\ \pm a_a^{-+} \mp ic_a & 1 \mp a_a^{+-} \end{pmatrix} \tag{40}$$

$$\mathbf{A}^{\pm}_{a,1,1} = \sum_{m,m'=-1}^{+1} |m, X\rangle_{1,1} A^{X}_{m,m'} \langle m', X|_{1,1} = \frac{1}{2}\begin{pmatrix} 1 \pm a_a^{-+} & \pm a_a^{+-} \mp ic_a \\ \pm a_a^{+-} \pm ic_a & 1 \mp a_a^{-+} \end{pmatrix} \tag{41}$$

The laboratory "Z" and "X" representations refer to the orthonormal states Eqs.(14) and (15) after a unitary transformation to the laboratory frame. As column vectors, they have the same form as Eqs.(16),

$$|+, Z\rangle_{n_z,n_x} \equiv \left|+, \sqrt{2}\right\rangle_{n_z,n_x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{n_z,n_x} ; |-, Z\rangle_{n_z,n_x} \equiv |-, \sqrt{2}\rangle_{n_z,n_x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{n_z,n_x} \tag{42}$$

$$|\pm, X\rangle_{n_z,n_x} \equiv T\,|\pm, \sqrt{2}\rangle_{n_z,n_x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}_{n_z,n_x} \tag{43}$$

but in the $|\pm, \sqrt{2}\rangle_{n_z,n_x}$ basis, rather than the $|\pm, z\rangle$ or $|\pm, x\rangle$ bases given in Eq.(16). The transformation between the Z and X representations is given by,

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_{n_z,n_x} \tag{44}$$

From Eqs.(40) and (41) the quantum coherence in the Z representation becomes eigenvalues in the X representation and vice versa. Another way of saying this is Nature hides half the correlation [40] from us because the act of measurement (*cf.* Figure 3). If we do not measure, a spin carries a magnetic moment of $\sqrt{2}\mu$ which is responsible for the quantum correlation and violation of Bell's Inequalities. If it is measured, then the usual spin is observed with magnetic moment of μ.

Repeating the simulation in exactly the same way, but with $a^{+-} \leftrightarrow a^{+-}$, shifts from the Z to the X representation but does not change Figure 4. Likewise changing all the filter angles by shifting them by $\frac{\pi}{2}$, or any other angle, gives identical results with correlation also fitting Eq.(39). Therefore half the correlation is obtained from the calculation in the Z representation and the other half from the X representation. The sum of the two gives the full quantum correlation from both axes, Eq.(1).

COHERENCE, FILTER SETTING AND CONSISTENCY WITH BELL'S INEQUALITIES

One argument in favor of the two axes model is the probabilities in Eq.(36) have the same form as the polarization displayed in Bell's inequalities, and predicts the filter settings that maximize their violation. From Eq.(37), the polarization of a two dimensional spin approaching a filter set at $\mathbf{a}$ is,

$$P_a^+ - P_a^- = a_a^{+-} \tag{45}$$

given for the even quadrants in Eq.(30). Using straightforward trigonometry, three vectors can be identified. In the Lab frame $(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}})$: one is the filter angle,

$$\mathbf{a} = \cos\theta_a \hat{\mathbf{Z}} + \sin\theta_a \cos\phi_a \hat{\mathbf{X}} + \sin\theta_a \sin\phi_a \hat{\mathbf{Y}} \tag{46}$$

and the other two depend only upon the LHV θ, ϕ and are orthogonal,

$$\mathbf{b} = \cos\theta \hat{\mathbf{Z}} + \sin\theta \cos\phi \hat{\mathbf{X}} + \sin\theta \sin\phi \hat{\mathbf{Y}} \tag{47}$$

$$\mathbf{c} = -\sin\theta \hat{\mathbf{Z}} + \cos\theta \cos\phi \hat{\mathbf{X}} + \cos\theta \sin\phi \hat{\mathbf{Y}} \tag{48}$$

and simplifies the polarization carried by a single spin to

$$a_a^{+-} = \frac{1}{\sqrt{2}}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) \tag{49}$$

$$a_a^{-+} = \frac{1}{\sqrt{2}}\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) \tag{50}$$

The polarization carried by a 2D spin has the same form as Bell's original Inequalities when the quantum correlation is obtained using a singlet state, [10],

$$\sqrt{2}\left|a_a^{+-}\right| = |\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})| \leq (1+\mathbf{b}\cdot\mathbf{c}) = 1 \tag{51}$$

$$\sqrt{2}\left|a_d^{-+}\right| = |\mathbf{d}\cdot(\mathbf{b}-\mathbf{c}| \leq (1-\mathbf{b}\cdot\mathbf{c}) = 1 \tag{52}$$

The vectors $\mathbf{b}$ and $\mathbf{c}$, being orthogonal, have magnitude of $\sqrt{2}$,

$$|\mathbf{b}\pm\mathbf{c}| = \sqrt{2} \tag{53}$$

Bell's original inequalities are maximally violated under two conditions: either $\mathbf{a}$ is collinear to $\mathbf{b}+\mathbf{c}$ or $\mathbf{d}$ is collinear to $\mathbf{b}-\mathbf{c}$, both give the violation $\sqrt{2}\leq 1$. Of course it is impossible to apply filters simultaneously in the $\mathbf{a}$ and $\mathbf{d}$ directions. Figure 5 depicts the vectors that maximize the correlation, and these agree with the filter settings that maximize the CHSH equation, Eq.(4).

The sum of the two terms in Eqs.(51) and (52) has the same form as the CHSH equation after the quantum results have been substituted. This displays the full quantum correlation when both coefficients $|a_a^{\pm\mp}| = 1$,

$$|\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})+\mathbf{d}\cdot(\mathbf{b}-\mathbf{c})| = \left|a_a^{+-}+a_d^{-+}\right|\sqrt{2} \to 2\sqrt{2} \tag{54}$$

However, the two polarizations $a_a^{\pm\mp}$ cannot both simultaneously attain unity in this model. Whereas Bell's original inequalities, Eqs.(51) and (52), are violated, the CHSH equation is not. For a given set of local hidden variables, LHV, λ, the CHSH derivation defines two regions expressed by,

$$B(\mathbf{b},\lambda)+B(\mathbf{c},\lambda) \tag{55}$$

$$B(\mathbf{b},\lambda)-B(\mathbf{c},\lambda) \tag{56}$$

where $\mathbf{b}$ and $\mathbf{c}$ are filter angle settings. Whereas the CHSH equations classically partition LHV space, in the treatment here the two orthogonal axes, Eqs.(55) and (56), make those regions experimentally complementary.

Using this local realistic model, the correlation between EPR pairs is indistinguishable from that obtained by assuming the persistence of non-local entanglement. The simulation using the 2D spin model, therefore, generates the quantum result showing that a LHV model can produce the EPR correlations without violating local causality.

Analysis of CHSH geometry

Further corroboration for two axes comes from the independent analysis of the CHSH equation by Gustafson [41]. He finds a vector within the CHSH equation that has maximum magnitude of $\sqrt{2}$ which is solely responsible for the quantum violation. The properties of this vector are identical to the properties of the 2D spin proposed here. The quantum correlation in the CHSH equation can be written in the following form, (*cf* Eq.(54)),

$$|\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})+\mathbf{d}\cdot(\mathbf{b}-\mathbf{c})| = 2\hat{\mathbf{u}}_1\cdot\mathbf{u}_2 = 2\left(\cos^2\theta_{a,b+c}+\cos^2\theta_{d,b-c}\right)^{1/2}\cos\theta_{u_1,u_2} \xrightarrow{\max} 2\sqrt{2} \tag{57}$$

where one the two vectors is normalized, $|\hat{\mathbf{u}}_1| = 1$,

$$\hat{\mathbf{u}}_1 = \begin{pmatrix} \cos\frac{\theta_{b,c}}{2} \\ \sin\frac{\theta_{b,c}}{2} \end{pmatrix} \tag{58}$$

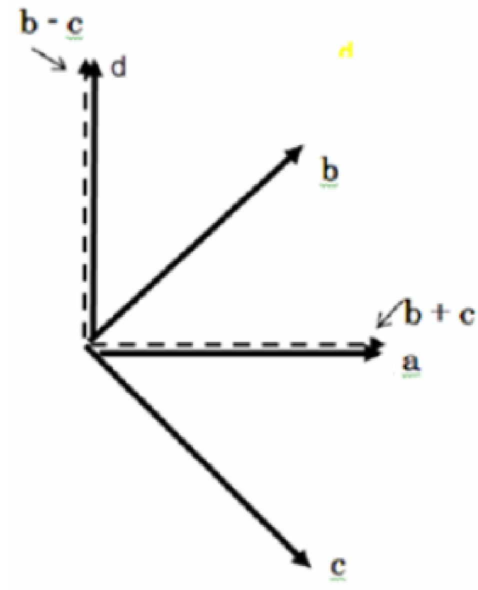

FIG. 5: The filter settings that maximize the correlation between an EPR pair. Note that the vector $\mathbf{a}$ is collinear to the sum $\mathbf{b}+\mathbf{c}$ and the vector $\mathbf{d}$ is collinear to the difference $\mathbf{b}-\mathbf{c}$. Compare with Figure 2.

and the other is unnormalized, $|\mathbf{u}_2| = \left(\cos^2\theta_{a,b+c} + \cos^2\theta_{d,b-c}\right)^{1/2} \to \sqrt{2}$,

$$\mathbf{u}_2 = \begin{pmatrix} \cos\theta_{a,b+c} \\ \cos\theta_{d,b-c} \end{pmatrix} \tag{59}$$

The maximum length of the vector $\mathbf{u_2}$ is $\sqrt{2}$ which requires that $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ be collinear as must simultaneously $\mathbf{d}$ and $\mathbf{b}-\mathbf{c}$. Finally to maximize the CHSH equation, $\cos\theta_{u_1,u_2} = 1$ meaning that $\hat{\mathbf{u}}_1$ and $\mathbf{u}_2$ must also be collinear. Applying these conditions agrees with the filter settings in Figure 5. In summary Gustafson's vector of length $\sqrt{2}$ has the same geometrical properties as the 2D spin presented here.

CONCLUSIONS

By assuming a single spin has two permanent and orthogonal magnetic moments of magnitude μ each, a locally realistic reconciliation of the EPR paradox follows. Each axis carries half the correlation and gives the quantum result of $-\cos\theta_{ab}$ when the two are summed. A 2D spin forms beable states that are superpositions of the two orthogonal axes of spin quantization, Eqs.(14) and (15). These states can only form in an isolated and isotropic environment because then the two axes are indistinguishable. A probe destroys these states.

The Heisenberg Uncertainty Principle plays a fundamental role in this treatment and is physically caused, in this case, by randomization of the quantum coherence terms. This leads to the conclusion that the CHSH Inequalities cannot be violated by this model. In contrast Bell's original inequalities, Eq.(51) are violated as predicted.

It is usually assumed that repeated acts of measurement can fully characterize the pure states of a system. This model suggests that there are beable properties which, in this case, can never be observed, and leads to the conclusion that properties can be missed by the act of measurement (in this case half the spin polarization). Since spin plays a fundamental role in Nature, missing properties due to decoherence might be a common occurrence.

The complementary nature of the two axes underscores the fact that there is no experimental way of confirming that spin has one axis of quantization, (a point particle), or two, (with structure). That would require counting coincidences that are counterfactual. However the simulation reproduces the quantum state consistent with the Statistical Ensemble Interpretation of quantum mechanics, [7]. Moreover, the consistency of this model with Bell's Inequalities is convincing evidence for two axes. Since coincidence experiment cannot resolve the issue, subjective arguments alone must decide. Besides the simulation agreeing with both quantum theory and experiment, additional support is provided by: the consistency with Bell's correlation; prediction of the filter settings that maximize their violation; and agreement with the treatment of Gustafson, [41].

Although only quantum concepts are used in this model, the operator set $(\sigma_x, i\sigma_y, \sigma_z)$ introduces differences within the structure of quantum theory that have not been discussed. In this model the operator set obeys the same $su(2)$ algebra as the usual spin, but quantum theory is applied here in the absence of a measuring probe. The current and popular view that quantum mechanics be interpreted as quantum information underscores the fact that information is obtained only by measurement. One conclusion here is that quantum mechanics needs to be extended to describe states beyond our ability to measure. The situation is also different when the model spins interact with each other [42] or with local fields, and this is deferred to future work.

Additional information

The derivations of all the equations and more information about the spin model used can be found at http://quantummechanics.mchmultimedia.com/

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